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CROSSED MODULES AND DOI–HOPF MODULES

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ABSTRACT

We prove that crossed modules (or Yetter–Drinfel'd modules) are special cases of Doi's unified Hopf modules. The category of crossed H-modules is therefore a Grothendieck category (if we work over a field), and the Drinfel'd double appears as a type of generalized smash product.

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0. Introduction

Let H be a Hopf algebra with bijective antipode over a commutative ring k. As is well-known, a Hopf module is a k-module that is at once an H-module and an H-comodule, with a certain compatibility relation; see [30] for details. Doi [11] generalized this concept in the following way: if A is an H-comodule algebra, and C is an H-module coalgebra, then he introduced a so-called unified Hopf module: this is a k-module that is at once an A-module and a C-comodule, satisfying a compatibility relation that is an immediate generalization of the one that may be found in Sweedler's book [30]. One of the nice features here is that Doi's Hopf modules (we will call them Doi–Hopf modules) really unify a lot of module structures that have been studied by several authors; let us mention Sweedler's Hopf modules [30], Takeuchi's relative Hopf modules [31], graded modules, and modules graded by a G–set. In [5], induction functors between categories of Doi–Hopf modules and their adjoints are studied, and it turns out that many pairs of adjoint functors studied in the literature (the forgetful functor and its adjoint, extension and restriction of scalars,...) are special cases.

A type of modules that seem to be of a different nature are the crossed H-modules, also called Yetter–Drinfel'd modules. These are modules which are at once H-modules and H-comodules, with a certain compatibility relation. The key idea is that the compatibility relation is such that, in the case where the Hopf algebra H is finite dimensional, the crossed H-modules are nothing else than modules over the Drinfel'd double $D(H)$, as introduced by Drinfel'd ([12]). This becomes clear if one views H^*-modules as H-comodules, and if one considers Majid's form of the Drinfel'd double ([20]). Crossed modules have been studied extensively by many authors; let us mention [28, 17]. In the special case where H is commutative and cocommutative, crossed H-modules are just dimodules, as studied by Long in [18]. If in addition H is finite, then a crossed H-module is nothing else than an $H \otimes H^*$-module, and a Hopf module in Sweedler's sense is then an $H\#H^*$-module. This illustrates of course the fact that we are considering two basically different types of modules.

However, there is a relation between crossed H-modules and Hopf modules. In a recent paper, P. Schauenburg ([29]) shows that the category of crossed H-modules is equivalent to the category of two-sided Hopf modules ${}_H^H\mathcal{M}_H^H$. The functor connecting the two categories is given by tensoring up to the left over H.

In this paper, we use a different approach. We can show that crossed modules appear as special cases of Doi–Hopf modules. To this end, we proceed as follows. First, we generalize the notion of crossed module following Doi's philosophy: if A is an H-bicomodule algebra and C is an H-bimodule coalgebra, then we can introduce the notion of (H, A, C)-crossed H-module. These are modules that are at once A-modules and C-comodules such that a certain compatibility relation holds. We can show that $(H^{\rm op}\otimes H, A, C)$ is then a Doi–Hopf datum, and our main result is that the categories of (H, A, C)-crossed H-modules and $(H^{\rm op}\otimes H, A, C)$-Doi–Hopf modules are isomorphic (cf. Theorem 2.3). Of course, if $H = A = C$, then we obtain a description of the category of "classical" crossed H-modules.

As an application, we can give induction functors and their adjoints between categories of crossed H-modules. It suffices to translate the results of [5] to our actual situation; this is done in Proposition 2.6. In particular, we obtain results about the functors forgetting the module or comodule structure in the category of crossed H-modules; see Corollaries 2.8 and 2.10. Another consequence is that the category of crossed H-modules is a Grothendieck category, at least if we work over a field.

In Section 3, we focus attention on the Drinfel'd double. We introduce the generalized Drinfel'd double $D_H(A, C)$, and we show that Majid's Drinfel'd double is just $D_H(H, H)$. If C is finite, then the category of Yetter–Drinfel'd modules is isomorphic to the category of $D_H(A, C)$-modules.

Majid has focused attention on the following problem: when can we write the Drinfel'd double as a smash product? For a general Hopf algebra, he introduced a "double crossed product" $\bowtie$ of two Hopf algebras ([19]), and he could show that $D(H)$ is an example of such a double cross product. If H is quasitriangular, then Majid ([20]) shows that $D(H)$ is a usual smash product of an H-module algebra and a Hopf algebra. From our theory, it follows that the Drinfel'd double is a generalized type of smash product, as introduced by Takeuchi in [32].

In an Appendix, we have collected some basic results about the smash coproduct of an an H-module coalgebra D and an H-comodule coalgebra C. The proofs of most of the results are dual analogues of proofs of corresponding properties of the smash product, and this is why we omitted them. In fact, if both C and D are finitely generated and projective, then the smash coproduct is the dual of the smash product of the duals. In particular, the dual of the Drinfel'd double (the Drinfel'd codouble) appears as a generalized smash coproduct. The referee kindly informed us that a description of the Drinfel'd codouble was given earlier by Majid in [21, p. 297 and p. 318].

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1. Preliminaries

Throughout this paper, k will be a commutative ring. At some places, we will need that k is a field, and then we will mention this explicitly. Unless specified otherwise, all modules, algebras, coalgebras and Hopf algebras that we consider are over k. $\otimes$ and Hom will mean $\otimes_k$ and Hom_k. For a coalgebra C, we will use Sweedler's Σ-notation, that is, $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$, $(I \otimes \Delta)\Delta(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$, etc. We will also use the Sweedler notation for left and right C-comodules: $\rho_M(m) = \sum m_{[0]} \otimes m_{[1]}$, or $\rho_M(m) = \sum m_{\{0\}} \otimes m_{\{1\}}$ for any $m \in M$ if (M, ρ_M) is a right C-comodule and $\rho_N(n) = \sum n_{[-1]} \otimes n_{[0]}$, or $\rho_N(n) = \sum n_{\{-1\}} \otimes n_{\{0\}}$ for any n if (N, ρ_N) is a left C-comodule. $\mathcal{M}^C$ will be the category of right C-comodules and C-colinear maps and ${}_A\mathcal{M}$ will be the category of left A-modules and A-linear maps, if A is a k-algebra.

For two modules V and W, $\tau: V \otimes W \to W \otimes V$ will denote the switch map, that is, $\tau(v \otimes w) = w \otimes v$ for all $v \in V$ and $w \in W$.

Recall that a left H-module algebra is an algebra D which is also a left H-module such that $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$ and $h \cdot 1_D = \varepsilon(h) 1_D$, for all $h \in H$, $a, b \in D$. A right (left) H-comodule algebra is an algebra A which is also a right (left) H-comodule such that the structure map $\rho_A: A \to A \otimes H$, $\rho_A(a) = \sum a_{[0]} \otimes a_{[1]}$ ($\rho_A: A \to H \otimes A$, $\rho_A(a) = \sum a_{[-1]} \otimes a_{[0]}$) is an algebra map. A is called an H-bicomodule algebra if A is an H-bicomodule which is also a left and right H-comodule algebra. Dually, C is a right (left) H-module coalgebra if C is a coalgebra which is also a right (left) H-module such that the structure map $C \otimes H \to C$, $c \otimes h \mapsto (c \leftharpoonup h)$ (resp. $H \otimes C \to C$, $h \otimes c \mapsto (h \rightharpoonup c)$) is a coalgebra map. C is called an H-bimodule coalgebra if C is a coalgebra which is also an H-bimodule such that C is a left and right H-module coalgebra.

A coalgebra C is a right H-comodule coalgebra if C is a right H-comodule via $\rho_C: C \to C \otimes H$, $\rho_C(c) = \sum c_{[0]} \otimes c_{[1]}$ such that

$$\sum c_{[0](1)} \otimes c_{[0](2)} \otimes c_{[1]} = \sum c_{(1)[0]} \otimes c_{(2)[0]} \otimes c_{(1)[1]} c_{(2)[1]}$$

and

$$\sum \varepsilon_C(c_{[0]})c_{[1]} = \varepsilon_C(c)1_H$$

for all $c \in C$.

Let A be a right H-comodule algebra and C be a left H-module coalgebra. In the sequel, we will call the threetuple (H, A, C) a Doi–Hopf datum. A left-right (H, A, C)-Hopf module is a k-module M, which is a left A-module and a right C-comodule via $\rho_M\colon M \to M \otimes C$ such that

$$\rho_M(am) = \sum a_{[0]}m_{[0]} \otimes a_{[1]} \rightharpoonup m_{[1]},$$

for all $a \in A$, $m \in M$. ${}_A\mathcal{M}(H)^C$ will be the category of left-right (H, A, C)-Hopf modules and A-linear C-colinear homomorphisms (see [11]). Similarly, if A is a left H-comodule algebra and C is a right H-module coalgebra we can construct ${}^C\mathcal{M}(H)_A$ (see [11]).

Now consider another Doi–Hopf datum (H', A', C'). A morphism

$$F = (f, u, v)\colon (H, A, C) \to (H', A', C')$$

between (H, A, C) and (H', A', C') consists of a threetuple (f, u, v), where $f\colon H \to H'$ is a Hopf algebra map, $u\colon A \to A'$ is a morphism of right H'-comodule algebras (A is a right H'-comodule algebra via f) and $v\colon C \to C'$ is a morphism of left H-module coalgebras (C' is a left H-module coalgebra via f). The following is just the left-right variant of Theorem 1.3 of [5].

THEOREM 1.1 ([5]): *With the above assumption, we have two functors*

$${}_A\mathcal{M}(H)^C \overset{F^*}{\to} {}_{A'}\mathcal{M}(H')^{C'} \overset{F_*}{\to} {}_A\mathcal{M}(H)^C$$

defined as follows. For $M \in {}_A\mathcal{M}(H)^C$, $F^*(M) = A' \otimes_A M \in {}_{A'}\mathcal{M}(H')^{C'}$ with A'-action and C'-coaction given by

$$a' \cdot (b' \otimes_A m) = a'b' \otimes_A m,$$
$$\rho_{F^*(M)}(a' \otimes_A m) = \sum (a'_{[0]} \otimes_A m_{[0]}) \otimes (a'_{[1]} \rightharpoonup v(m_{[1]})),$$

for all $a', b' \in A'$, $m \in M$.

For $N \in {}_{A'}\mathcal{M}(H')^{C'}$, $F_*(N) = N \square_{C'} C \in {}_A\mathcal{M}(H)^C$ with A'-action and C'-coaction given by

$$a \cdot (\sum n_i \otimes c_i) = \sum u(a_{[0]})n_i \otimes a_{[1]} \rightharpoonup c_i,$$
$$\rho_{F_*(N)}(\sum n_i \otimes c_i) = \sum n_i \otimes c_{i(1)} \otimes c_{i(2)},$$

for all $a \in A$ *and* $\sum n_i \otimes c_i \in N_{\square_{C'}} C$.

Moreover, the functor F_* *is a right adjoint of* F^*. *In particular, the forgetful functor* U^C: ${}_A\mathcal{M}(H)^C \to {}_A\mathcal{M}$ *has a right adjoint* $\bullet \otimes C$: ${}_A\mathcal{M} \to {}_A\mathcal{M}(H)^C$.

Observe that $A \otimes C$ is an object of ${}_A\mathcal{M}(H)^C$. The structure maps are the following:

$$a \cdot (b \otimes c) = \sum a_{[0]} b \otimes a_{[1]} \rightharpoonup c; \quad \rho_{A \otimes C}(b \otimes c) = \sum b \otimes c_{(1)} \otimes c_{(2)}$$

for all $a, b \in A$, $c \in C$.

THEOREM 1.2 ([22, Theorem 5]): *The functor* $\bullet \otimes C$: ${}_A\mathcal{M} \to {}_A\mathcal{M}(H)^C$ *has also a right adjoint*

$$\mathrm{Hom}_A^C(A \otimes C, \bullet): {}_A\mathcal{M}(H)^C \to {}_A\mathcal{M}$$

defined as follows: for $M \in {}_A\mathcal{M}(H)^C$, *the left action of* A *on* $\mathrm{Hom}_A^C(A \otimes C, M)$ *is given by the formula*

$$(a \cdot f)(b \otimes c) = f(ba \otimes c),$$

for all $a, b \in A$ *and* $c \in C$.

Recall that a left-right crossed H-module (or a Yetter–Drinfel'd module) is a k-module, which is at once a left H-module and a right H-comodule, such that the following compatibility relation holds:

$$\sum h_{(1)} \cdot m_{[0]} \otimes h_{(2)} m_{[1]} = \sum (h_{(2)} \cdot m)_{[0]} \otimes (h_{(2)} \cdot m)_{[1]} h_{(1)} \tag{1}$$

for all $h \in H$ and $m \in M$. If the antipode of H is bijective, (1) is equivalent to

$$\rho_M(h \cdot m) = \sum h_{(2)} \cdot m_{[0]} \otimes h_{(3)} m_{[1]} S^{-1}(h_{(1)}) \tag{2}$$

(see [17, Lemma 5.1.1]). The category of left-right Yetter–Drinfel'd modules and H-linear H-colinear maps will be denoted by ${}_H\mathcal{D}^H$. In a similar way, we can introduce left-left, right-right and right-left crossed H-modules. The corresponding categories are ${}_H^H\mathcal{D}$, $\mathcal{D}_H^H$ and ${}^H\mathcal{D}_H$; we refer to [28] for full details. If H is finite dimensional, then the categories $D(H)$-mod and ${}_H\mathcal{D}^H$ are isomorphic.

If $H = kG$, then an H-crossed module (or G-crossed module) is a G-graded module M with a G-action such that

$$\deg(\sigma \cdot m) = \sigma \deg(m) \sigma^{-1}$$

for all $\sigma \in G$ and $m \in M$ homogeneous. Now let $X = G$ with G-action $\sigma \cdot \tau = \sigma\tau\sigma^{-1}$. Then X is a left G-set, and it is clear that a module graded by X in the sense of [26] is the same thing as a G-crossed module (see also [16, Exercice 7, p. 236]). Now modules graded by G-sets are special cases of Doi–Hopf modules (cf. [11]), so crossed G-sets are Doi–Hopf modules. In the next Section, we will show that this property holds for arbitrary Hopf algebras.

2. Crossed H-modules versus Doi–Hopf modules

Let H be a Hopf algebra with invertible antipode, A an H-bicomodule algebra, and C an H-bimodule coalgebra. We denote the H-coactions on C by

$$\rho^{\ell}(a) = \sum a_{[-1]} \otimes a_{[0]} \quad \text{and} \quad \rho^{r}(a) = \sum a_{[0]} \otimes a_{[1]}$$

for all $a \in A$. For $h \in H$ and $c \in C$, we write $h \rightharpoonup c$ and $c \leftharpoonup h$ for the left and right action of h on c. In this situation, we will call the threetuple $G = (H, A, C)$ a **Yetter–Drinfel'd datum** or, in short, a **Drinfel'd datum**.

Let $G = (H, A, C)$ and $G' = (H', A', C')$ be two Yetter–Drinfel'd data. A threetuple $F = (f, u, v)$: $G \to G'$ is called a **morphism of Yetter–Drinfel'd data** if f: $H \to H'$ is a Hopf algebra map, u: $A \to A'$ is a morphism of H'-bicomodule algebras (A is an H'-bicomodule via f) and v: $C \to C'$ is a morphism of H-bimodule coalgebras (C' is an H-bimodule via f).

Given a Drinfel'd datum $G = (H, A, C)$, we can define four different types of crossed modules. We have a left-right, right-left, right-right and left-left version. A **crossed** (H, A, C)**-module** is a k-module M that is at once a (left or right) C-comodule and a (left or right) A-module such that

$$\rho_M(a \cdot m) = \sum a_{[0]} \cdot m_{[0]} \otimes a_{[1]} \rightharpoonup m_{[1]} \leftharpoonup S^{-1}(a_{[-1]}) \tag{3}$$
(left-right version),

$$\rho_M(m \cdot a) = \sum S^{-1}(a_{[1]}) \rightharpoonup m_{[-1]} \leftharpoonup a_{[-1]} \otimes m_{[0]} \cdot a_{[0]} \tag{4}$$
(right-left version),

$$\rho_M(m \cdot a) = \sum m_{[0]} \cdot a_{[0]} \otimes S(a_{[-1]}) \rightharpoonup m_{[1]} \leftharpoonup a_{[1]} \tag{5}$$
(right-right version),

$$\rho_M(a \cdot m) = \sum a_{[-1]} \rightharpoonup m_{[-1]} \leftharpoonup S(a_{[1]}) \otimes a_{[0]} \cdot m_{[0]} \tag{6}$$
(left-left version).

The four categories of crossed (H,A,C)-modules and A-linear, C-colinear maps are denoted by ${}_A\mathcal{D}(H)^C$, ${}^C\mathcal{D}(H)_A$, $\mathcal{D}(H)_A^C$ and ${}_A^C\mathcal{D}(H)$.

It follows easily that, for $C = A = H$, we obtain the classical crossed modules that we recalled at the end of Section 1.

We will next show that there exist relationships between the four types of crossed H-modules introduced above. First observe that if $G = (H,A,C)$ is a Drinfel'd datum, then $G' = (H^{\mathrm{op\,cop}}, A^{\mathrm{op}}, C^{\mathrm{cop}})$ is also a Drinfel'd datum. The $H^{\mathrm{op\,cop}}$-bicomodule structure on A^{op} and the $H^{\mathrm{op\,cop}}$-bimodule structure on C^{cop} are given by the following formulas

$$
\begin{aligned}
A^{\mathrm{op}} &\to H^{\mathrm{op\,cop}} \otimes A^{\mathrm{op}}\colon a \mapsto \sum a_{\{-1\}} \otimes a_{\{0\}} := \sum a_{[1]} \otimes a_{[0]},\\
A^{\mathrm{op}} &\to A^{\mathrm{op}} \otimes H^{\mathrm{op\,cop}}\colon a \mapsto \sum a_{\{0\}} \otimes a_{\{1\}} := \sum a_{[0]} \otimes a_{[1]},\\
H^{\mathrm{op\,cop}} \otimes C^{\mathrm{cop}} &\to C^{\mathrm{cop}}\colon h \otimes c \mapsto (h \rightharpoondown c) := c \leftharpoonup h,\\
C^{\mathrm{cop}} \otimes H^{\mathrm{op\,cop}} &\to C^{\mathrm{cop}}\colon c \otimes h \mapsto (c \leftharpoondown h) := h \rightharpoonup c,
\end{aligned}
$$

for all $a \in A^{\mathrm{op}}$, $h \in H^{\mathrm{op\,cop}}$, $c \in C^{\mathrm{cop}}$. It follows that

$$h \rightharpoondown c \leftharpoondown k = k \rightharpoonup c \leftharpoonup h, \tag{7}$$

for all $c \in C$, $h, k \in H$. Now we can easily show the following:

PROPOSITION 2.1: *Let $G = (H,A,C)$ be a Drinfel'd datum.*

(1) *The categories ${}^C\mathcal{D}(H)_A$ and ${}_{A^{\mathrm{op}}}\mathcal{D}(H^{\mathrm{op\,cop}})^{C^{\mathrm{cop}}}$ are isomorphic.*
(2) *The categories $\mathcal{D}(H)_A^C$ and ${}_{A^{\mathrm{op}}}^{C^{\mathrm{cop}}}\mathcal{D}(H^{\mathrm{op\,cop}})$ are isomorphic.*

Proof: (1) Take $(M, \cdot, \rho_M) \in {}^C\mathcal{D}(H)_A$. Then M is a left A^{op}-module with left action given by $a \triangleright m := m \cdot a$ and a right C^{cop}-comodule with coaction given by

$$\tilde{\rho}_M\colon M \to M \otimes C^{\mathrm{cop}}\colon m \mapsto \sum m_{\{0\}} \otimes m_{\{1\}} = \sum m_{[0]} \otimes m_{[-1]}.$$

A straightforward computation shows that $(M, \triangleright, \tilde{\rho}_M) \in {}_{A^{\mathrm{op}}}\mathcal{D}(H^{\mathrm{op\,cop}})^{C^{\mathrm{cop}}}$.

(2) Let $(M, \cdot, \rho_M) \in \mathcal{D}(H)_A^C$. Then M is a left A^{op}-module with action given by $a \triangleright m := m \cdot a$, and a left C^{cop}-comodule with coaction given by

$$\tilde{\rho}_M\colon M \to C^{\mathrm{cop}} \otimes M\colon m \mapsto \sum m_{\{-1\}} \otimes m_{\{0\}} = \sum m_{[1]} \otimes m_{[0]}.$$

It is easy to prove that $M \in {}_{A^{\mathrm{op}}}^{C^{\mathrm{cop}}}\mathcal{D}(H^{\mathrm{op\,cop}})$. ∎

PROPOSITION 2.2: *Let $G = (H, A, C)$ be a Drinfel'd datum. Then*

(1) *$G' = (H, A, C^{\rm cop})$ is also a Drinfel'd datum. We keep the coaction of H on A, and modify the action of H on $C^{\rm cop}$ as follows:*

$$c \leftharpoondown h = S^{-1}(h) \rightharpoonup c \quad \text{and} \quad h \rightharpoondown c = c \leftharpoonup S^{-1}(h) \tag{8}$$

for all $h \in H$, $c \in C^{\rm cop}$;

(2) *the categories ${}^{C}\mathcal{D}(H)_A$ and $\mathcal{D}(H)_A^{C^{\rm cop}}$ are isomorphic;*

(3) *the categories ${}_A\mathcal{D}(H)^C$ and ${}_A^{C^{\rm cop}}\mathcal{D}(H)$ are isomorphic.*

Proof: (1) It is obvious that $C^{\rm cop}$ with the actions given by (8) is an H-bimodule. We will prove that $C^{\rm cop}$ is an H-bimodule coalgebra. For all $c \in C^{\rm cop}$ and $h \in H$, we have

$$\begin{aligned} \Delta_{C^{\rm cop}}(c \leftharpoondown h) &= \Delta_{C^{\rm cop}}(S^{-1}(h) \rightharpoonup c) \\ (\rightharpoonup \text{ is a coalgebra map}) &= \sum (S^{-1}(h_{(1)}) \rightharpoonup c_{(2)}) \otimes (S^{-1}(h_{(2)}) \rightharpoonup c_{(1)}) \\ &= \sum (c_{(2)} \leftharpoondown h_{(1)}) \otimes (c_{(1)} \leftharpoondown h_{(2)}), \end{aligned}$$

and it follows that $(C^{\rm cop}, \leftharpoondown)$ is a right H-module coalgebra. In a similar way, we obtain that

$$\Delta_{C^{\rm cop}}(h \rightharpoondown c) = \sum (h_{(1)} \rightharpoondown c_{(2)}) \otimes (h_{(2)} \rightharpoondown c_{(1)})$$

and this implies that $(C^{\rm cop}, \rightharpoondown)$ is a left H-module coalgebra.

From (8), it also follows that

$$S(h) \rightharpoondown c \leftharpoondown k = S^{-1}(k) \rightharpoonup c \leftharpoonup h \quad \text{and} \quad h \rightharpoondown c \leftharpoondown S(k) = k \rightharpoonup c \leftharpoonup S^{-1}(h) \tag{9}$$

for all $h, k \in H$, $c \in C$.

(2) Let $(M, \cdot, \rho_M) \in {}^{C}\mathcal{D}(H)_A$. M can be given the structure of a right $C^{\rm cop}$-comodule as follows:

$$\tilde{\rho}_M \colon M \to M \otimes C^{\rm cop}, \quad \tilde{\rho}_M(m) := \sum m_{\{0\}} \otimes m_{\{1\}} = \sum m_{[0]} \otimes m_{[-1]}$$

for all $m \in M$. It is straightforward to show that $(M, \cdot, \tilde{\rho}_M) \in \mathcal{D}(H)_A^{C^{\rm cop}}$.

(3) Take $(M, \cdot, \rho_M) \in {}_A\mathcal{D}(H)^C$. Then M is a left $C^{\rm cop}$-comodule via

$$\tilde{\rho}_M \colon M \to C^{\rm cop} \otimes M, \quad \tilde{\rho}_M(m) = \sum m_{\{-1\}} \otimes m_{\{0\}} = \sum m_{[1]} \otimes m_{[0]}.$$

We keep the original action of A on M. We leave it to the reader to show that $(M, \cdot, \tilde{\rho}_M) \in {}_A^{C^{\rm cop}}\mathcal{D}(H)$. ∎

We are now able to prove the main result of this paper.

THEOREM 2.3: *Let $G = (H, A, C)$ be a Drinfel'd datum.*

(1) *A can be made into a right $H^{\mathrm{op}} \otimes H$-comodule algebra. The coaction of $H^{\mathrm{op}} \otimes H$ on A is given by the map*

$$\rho_A: A \to A \otimes H^{\mathrm{op}} \otimes H: a \mapsto \sum a_{[0]} \otimes S^{-1}(a_{[-1]}) \otimes a_{[1]}.$$

(2) *C can be made into a left $H^{\mathrm{op}} \otimes H$-module coalgebra. The action of $H^{\mathrm{op}} \otimes H$ on C is given by the formula*

$$(h \otimes k) \triangleright c := k \rightharpoonup c \leftharpoonup h.$$

(3) *The categories ${}_A\mathcal{D}(H)^C$ and ${}_A\mathcal{M}(H^{\mathrm{op}} \otimes H)^C$ are isomorphic.*

Consequently, if k is a field, then ${}_A\mathcal{D}(H)^C$ is a Grothendieck category.

Proof: (1) Let us first prove that A is a right $H^{\mathrm{op}} \otimes H$-comodule. For all $a \in A$, we have

$$\begin{aligned}
&(I \otimes \Delta_{H^{\mathrm{op}} \otimes H})\rho_A(a) \\
&= \sum a_{[0]} \otimes \Delta_{H^{\mathrm{op}} \otimes H}(S^{-1}(a_{[-1]}) \otimes a_{[1]}) \\
&= \sum a_{[0]} \otimes S^{-1}(a_{[-1](2)}) \otimes a_{[1](1)} \otimes S^{-1}(a_{[-1](1)}) \otimes a_{[1](2)} \\
&= \sum a_{[0]} \otimes S^{-1}(a_{[-1]}) \otimes a_{[1]} \otimes S^{-1}(a_{[-2]}) \otimes a_{[2]} \\
&\qquad (A \text{ is } H\text{-bicomodule algebra}) \\
&= \sum \rho_A(a_{[0]}) \otimes S^{-1}(a_{[-1]}) \otimes a_{[1]} \\
&= (\rho_A \otimes I)\rho_A(a)
\end{aligned}$$

and therefore A is a right $H^{\mathrm{op}} \otimes H$-comodule.

We also have that

$$\begin{aligned}
\rho_A(ab) &= \sum a_{[0]}b_{[0]} \otimes S^{-1}(a_{[-1]}b_{[-1]}) \otimes a_{[1]}b_{[1]} \\
&= \sum a_{[0]}b_{[0]} \otimes S^{-1}(a_{[-1]}) \cdot S^{-1}(b_{[-1]}) \otimes a_{[1]}b_{[1]} \\
&= \sum (a_{[0]} \otimes S^{-1}(a_{[-1]}) \otimes a_{[1]})(b_{[0]} \otimes S^{-1}(b_{[-1]}) \otimes b_{[1]}) \\
&= \rho_A(a)\rho_A(b)
\end{aligned}$$

for all $a, b \in A$ ($\cdot$ is the product in H^{op}). This shows that A is a right $H^{\mathrm{op}} \otimes H$-comodule algebra.

(2) We will first prove that C is a left $H^{\mathrm{op}} \otimes H$-module. For all $h, k, l, m \in H$

and $c \in C$ we have that

$$\begin{aligned}(h \otimes k) \triangleright [(l \otimes m) \triangleright c] &= (h \otimes k) \triangleright (m \rightharpoonup c \leftharpoonup l)\\ &= k \rightharpoonup m \rightharpoonup c \leftharpoonup l \leftharpoonup h\\ &= km \rightharpoonup c \leftharpoonup lh\\ &= (lh \otimes km) \triangleright c\\ &= [(h \otimes k) \cdot (l \otimes m)] \triangleright c\end{aligned}$$

and this implies that C is a left $H^{\rm op} \otimes H$-module.

Using the fact that C is an H-bimodule algebra, we obtain that

$$\begin{aligned}\Delta_C((h \otimes k) \triangleright c) &= \Delta_C(k \rightharpoonup c \leftharpoonup h)\\ &= \sum (k_{(1)} \rightharpoonup c_{(1)} \leftharpoonup h_{(1)}) \otimes (k_{(2)} \rightharpoonup c_{(2)} \leftharpoonup h_{(2)})\\ &= \sum (h_{(1)} \otimes k_{(1)}) \triangleright c_{(1)} \otimes (h_{(2)} \otimes k_{(2)}) \triangleright c_{(2)},\end{aligned}$$

for all $h, k \in H$ and $c \in C$, and this means that C is a left $H^{\rm op} \otimes H$-module coalgebra.

(3) Let $(M, \cdot, \rho_M)$ be such that $(M, \cdot)$ is a left A-module and (M, ρ_M) is a right C-comodule. Then $M \in {}_A\mathcal{M}(H^{\rm op} \otimes H)^C$ if and only if

$$\begin{aligned}\rho_M(a \cdot m) &= \sum a_{[0]} \cdot m_{[0]} \otimes (S^{-1}(a_{[-1]}) \otimes a_{[1]}) \triangleright m_{[1]}\\ &= \sum a_{[0]} \cdot m_{[0]} \otimes a_{[1]} \rightharpoonup m_{[1]} \leftharpoonup S^{-1}(a_{[-1]})\end{aligned}$$

for all $a \in A$ and $m \in M$. This implies that $M \in {}_A\mathcal{M}(H^{\rm op} \otimes H)^C$ if and only if (3) holds, that is $M \in {}_A\mathcal{D}(H)^C$. This shows that ${}_A\mathcal{D}(H)^C$ is isomorphic to ${}_A\mathcal{M}(H^{\rm op} \otimes H)^C$.

The last statement now follows from the fact that the category of (H, A, C)-Hopf modules is always a Grothendieck category, at least if we work over a field (see [5]). ∎

If we apply Theorem 2.3 in the case where $C = A = H$, then we obtain the following:

COROLLARY 2.4: *Let H be a Hopf algebra with bijective antipode.*

(1) *H can be made into a right $H^{\rm op} \otimes H$-comodule algebra. The coaction $H \to H \otimes H^{\rm op} \otimes H$ is given by the formula*

$$h \mapsto \sum h_{(2)} \otimes S^{-1}(h_{(1)}) \otimes h_{(3)}.$$

(2) *H can be made into a left $H^{\mathrm{op}}\otimes H$-module coalgebra. The action of $H^{\mathrm{op}}\otimes H$ on H is given by the formula*

$$(h \otimes k) \triangleright l := klh.$$

(3) *The category ${}_H\mathcal{D}^H$ of left-right crossed H-modules is isomorphic to a category of Doi–Hopf modules, namely ${}_H\mathcal{M}(H^{\mathrm{op}}\otimes H)^H$.*

If k is a field then the categories ${}_H\mathcal{D}^H$, $\mathcal{D}_H^H$, ${}_H^H\mathcal{D}$ and ${}^H\mathcal{D}_H$ are Grothendieck categories.

Remark 2.5: Of course, Theorem 2.3 also holds for the categories ${}^C\mathcal{D}(H)_A$, $\mathcal{D}(H)_A^C$ and ${}_A^C\mathcal{D}(H)$. It suffices to apply Propositions 2.1 and 2.2.

For example, if $G = (H, A, C)$ is a Drinfel'd datum then A is a left $H \otimes H^{\mathrm{op}}$-comodule algebra in the following way:

$$a \mapsto \sum a_{[-1]} \otimes S^{-1}(a_{[1]}) \otimes a_{[0]}. \tag{10}$$

In a similar way, C can be given the structure of right $H \otimes H^{\mathrm{op}}$-module coalgebra structure:

$$c \cdot (h \otimes k) = k \rightharpoonup c \leftharpoonup h. \tag{11}$$

The categories ${}^C\mathcal{D}(H)_A$ and ${}^C\mathcal{M}(H \otimes H^{\mathrm{op}})_A$ are then isomorphic.

In Theorem 1.1 we have seen how we can construct an induction functor and its right adjoint between two categories of Doi–Hopf modules. Using Theorem 2.3, we can now apply this result to find pairs of adjoint functors between categories of crossed H-modules. Consider a morphism $F = (f, u, v)$: $G \to G'$ of Drinfel'd data. The right and left action of $h' \in H'$ on $c' \in C'$ will be denoted by $c' \leftharpoonup h'$ and $h' \rightharpoonup c'$.

It is easy to see that $f \otimes f$: $H^{\mathrm{op}} \otimes H \to H'^{\mathrm{op}} \otimes H'$ is a Hopf algebra map, that u: $A \to A'$ is a morphism of right $H'^{\mathrm{op}} \otimes H'$-comodule algebras and that v: $C \to C'$ is a morphism of left $H^{\mathrm{op}} \otimes H$-module coalgebras. From Theorem 1.1 it therefore follows that we have a pair of adjoint functors

$${}_A\mathcal{M}(H^{\mathrm{op}} \otimes H)^C \cong {}_A\mathcal{D}(H)^C \underset{F_*}{\overset{F^*}{\rightleftarrows}} {}_{A'}\mathcal{D}(H')^{C'} \cong {}_{A'}\mathcal{M}(H'^{\mathrm{op}} \otimes H')^{C'}.$$

The functors F^* and F_* may be described explicitly; it suffices to translate the formulas of Theorem 1.1 to our particular situation. We then obtain the following:

For $(M, \cdot, \rho_M) \in {}_A\mathcal{D}(H)^C$ we have that $F^*(M) = A' \otimes_A M$, where A' is right A-module via u: $A \to A'$. The A'-action and C'-coaction on $F^*(M)$ are given by the formulas

$$a' \triangleright (b' \otimes_A m) = a'b' \otimes_A m, \tag{12}$$

$$\rho_{F^*(M)}(a' \otimes_A m) = \sum a'_{\{0\}} \otimes_A m_{[0]} \otimes (a'_{\{1\}} \rightharpoonup v(m_{[1]}) \leftharpoonup S^{-1}(a'_{\{-1\}})), \tag{13}$$

for all $a', b' \in A'$ and $m \in M$. For $(N, \cdot, \rho_N) \in {}_{A'}\mathcal{D}(H')^{C'}$, we have that $F_*(N) = N\Box_{C'}C$, where C is a C'-bicomodule via $v\colon C \to C'$. The structure maps on $F_*(N)$ are given by the following data:

$$a \triangleright (\sum m_i \otimes c_i) = \sum u(a_{[0]}) \cdot m_i \otimes (a_{[1]} \rightharpoonup c_i \leftharpoonup S^{-1}(a_{[-1]})), \tag{14}$$

$$\rho_{F_*(N)}(\sum m_i \otimes c_i) = \sum m_i \otimes c_{i(1)} \otimes c_{i(2)}, \tag{15}$$

for all $a \in A$ and $\sum m_i \otimes c_i \in N\Box_{C'}C$.

We summarize our results as follows:

PROPOSITION 2.6: *Let $F = (f, u, v)\colon (H, A, C) \to (H', A', C')$ be a morphism of Yetter–Drinfel'd data. Then the functors F^* and F_* defined above are well defined and F_* is a right adjoint of F^*.*

Remarks 2.7: (1) Consider the particular situation where $H' = H$, $A' = A$, $C' = k$, $f = I_H$, $u = I_A$ and $v = \varepsilon_C$. Then the functor F^*, turns out to be the functor $U^C\colon {}_A\mathcal{D}(H)^C \to {}_A\mathcal{M}$, that is, the forgetting the C-comodule structure. This functor has a right adjoint

$$\bullet \otimes C\colon {}_A\mathcal{M} \to {}_A\mathcal{D}(H)^C\colon N \mapsto N \otimes C.$$

The structure maps on $N \otimes C$ are the following:

$$a \cdot (n \otimes c) = \sum a_{[0]} \cdot n \otimes a_{[1]} \rightharpoonup c \leftharpoonup S^{-1}(a_{[-1]}), \tag{16}$$

$$\rho_{N\otimes C}(n \otimes c) = \sum n \otimes c_{(1)} \otimes c_{(2)}, \tag{17}$$

for all $a \in A$, $n \in N$, $c \in C$.

(2) In a similar way, we can show that the functor forgetting the module structure has a left adjoint: Take $C = C'$, $H = H'$, $A = k$ (with the trivial structure of H-comodule algebra), $f = I_H$, $u = \eta_{A'}$ and $v = I_C$ in Proposition 2.6. The functor

$$F_* = {}_{A'}U\colon {}_{A'}\mathcal{D}(H')^{C'} \longrightarrow \mathcal{M}^{C'}$$

is the functor forgetting the A'-module structure. Its left adjoint

$$F^* = A' \otimes \bullet\colon \mathcal{M}^{C'} \longrightarrow {}_{A'}\mathcal{D}(H')^{C'}$$

may be described as follows: $F^*(M) = A' \otimes M$ with the following A'-module and C'-comodule structure:

$$a' \triangleright (b' \otimes m) = a'b' \otimes m, \tag{18}$$

$$\rho_{A' \otimes M}(a \otimes m) = \sum (a'_{\{0\}} \otimes m_{[0]}) \otimes (a'_{\{1\}} \rightharpoonup m_{[1]} \leftharpoonup S^{-1}(a'_{\{-1\}})), \tag{19}$$

for all $a', b' \in A'$, $m \in M$.

Suppose that the Hopf algebra H has a bijective antipode. We can apply the above remarks to the case where $H = A = C$, and then we obtain the following result.

COROLLARY 2.8: *Let H be a Hopf algebra with a bijective antipode.*

(1) *The forgetful functor U^H: ${}_H\mathcal{D}^H \to {}_H\mathcal{M}$ has a right adjoint $\bullet \oslash H$: ${}_H\mathcal{M} \to {}_H\mathcal{D}^H$.*
(2) *The forgetful functor ${}_HU$: ${}_H\mathcal{D}^H \to \mathcal{M}^H$ has a left adjoint $H \otimes \bullet$: $\mathcal{M}^H \to {}_H\mathcal{D}^H$.*

Remarks 2.9: (1) The adjoint pairs of the above results yield canonical morphisms. If we consider for example the functor forgetting the H-coaction, then we obtain the following canonical morphisms:

$$\nu\colon 1_{{}_H\mathcal{D}^H} \to (\bullet \otimes H) \circ U^H, \quad \nu_N\colon N \to N \oslash H, \quad \nu_N(n) = \sum n_{[0]} \otimes n_{[1]}, \tag{20}$$

for all $n \in N$, $N \in_H \mathcal{D}^H$ and

$$\delta\colon U^H \circ (\bullet \otimes H) \to 1_{{}_H\mathcal{M}}, \quad \delta_M\colon M \otimes H \to M, \quad \delta_M(m \otimes h) = \varepsilon(h)m, \tag{21}$$

for all $h \in H$, $m \in M$, $M \in {}_H\mathcal{M}$.

For the functor forgetting the H-action, we obtain the following:

$$\nu\colon 1_{\mathcal{M}^H} \to_H U \circ (H \otimes \bullet), \quad \nu_M\colon M \to H \otimes M, \quad \nu_M(m) = 1_H \otimes m, \tag{22}$$

for all $m \in M$, $M \in \mathcal{M}^H$ and

$$\delta\colon (H \otimes \bullet) \circ_H U \to 1_{{}_H\mathcal{D}^H}, \quad \delta_N\colon H \otimes N \to N, \quad \delta_N(h \otimes n) = h \cdot n, \tag{23}$$

for all $h \in H$, $n \in N$, $N \in_H \mathcal{D}^H$.

(2) It follows from Corollary 2.8 that we can give $H \otimes H$ the structure of crossed H-module in two ways:

$$\text{(Type I):} \qquad \begin{cases} h \cdot (l \otimes k) = \sum h_{(2)} l \otimes h_{(3)} k S^{-1}(h_{(1)}) \\ \rho_{H \otimes H}(l \otimes h) = \sum l \otimes h_{(1)} \otimes h_{(2)} \end{cases}$$

or

$$\text{(Type II):} \qquad \begin{cases} h \cdot' (l \otimes k) = hl \otimes k \\ \rho'_{H\otimes H}(l \otimes h) = \sum l_{(2)} \otimes h_{(1)} \otimes l_{(3)}h_{(2)}S^{-1}(l_{(1)}). \end{cases}$$

It follows from Corollary 2.4 and Example 1 of [5] that, as objects in ${}_H\mathcal{D}^H$, $H\otimes H$ with the type I structure is isomorphic to $H \otimes H$ with the type II structure.

(3) Applying (3) of Theorem 2.3 and Theorem 1.2 we find that the functor $\bullet \otimes H$: ${}_H\mathcal{M} \to {}_H\mathcal{D}^H$ has also a right adjoint $\mathrm{Hom}_H^H(H \otimes H, \bullet)$: ${}_H\mathcal{D}^H \to {}_H\mathcal{M}$. For $M \in_H \mathcal{D}^H$, the left H-action of H on $\mathrm{Hom}_H^H(H\otimes H, M)$ is defined as follows:

$$(h \cdot f)(l \otimes k) = f(lh \otimes k)$$

for all $h, k, l \in H$ and $f \in \mathrm{Hom}_H^H(H \otimes H, M)$.

(4) In [4], it is shown that for a projective Hopf algebra H, the above functor $\bullet \otimes H$ is also a left adjoint of the forgetful functor U^H if and only if H is a Frobenius and unimodular Hopf algebra.

We recall that an object M of an abelian category $\mathcal{A}$ with AB3 is called small if the functor $\mathrm{Hom}_{\mathcal{A}}(M, \bullet)$: $\mathcal{A} \to \underline{\underline{\mathrm{Ab}}}$ preserves direct sums (here $\underline{\underline{\mathrm{Ab}}}$ is the category of abelian groups). A coalgebra C is called right quasicofrobenius if there exists a right C^*-linear monomorphism from C to a free right C^*-module (see [14]). This is equivalent to C being a projective object of ${}^C\mathcal{M}$, cf. [15, Theorem 1.3]. If C is right quasicofrobenius then C is a generator of $\mathcal{M}^C$, cf. [14, 2.5]. A left and right quasicofrobenius coalgebra C is called quasicofrobenius.

COROLLARY 2.10: *Let H be a Hopf algebra with a bijective antipode.*

1. *The functor $\bullet\otimes H$: ${}_H\mathcal{M} \to {}_H\mathcal{D}^H$ is exact, commutes with direct sums, and preserves injective objects.*
2. *The functor $\bullet \otimes H$: ${}_H\mathcal{M} \to {}_H\mathcal{D}^H$ preserves cogenerators. In particular, $\mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Q}/\mathbb{Z}) \otimes H$ is an injective cogenerator of ${}_H\mathcal{D}^H$.*
3. *$(M, \cdot, \rho_M)$ is a small object of ${}_H\mathcal{D}^H$ if and only if $(M, \cdot)$ is a small left H-module.*
4. *The functor $H \otimes \bullet$: $\mathcal{M}^H \to {}_H\mathcal{D}^H$ preserves generators. In particular, if H is a quasicofrobenius coalgebra, then $H\otimes H$ is a projective generator of ${}_H\mathcal{D}^H$.*

Proof: (1) follows immediately from general properties of pairs of adjoint functors.

(2) $\bullet \otimes H$ is a right adjoint and the canonical morphism given by adjoints in (20) is a monomorphism, hence $\bullet\otimes H$ preserves cogenerators, cf. [23, lemma 2.7].

(3) U^H is a left adjoint of $\bullet \otimes H$ and U^H commutes with direct sums. Therefore U^H preserves small objects, cf. [14, Theorem 1.3]. Hence any small object M of ${}_H\mathcal{D}^H$ is also small as a left H-module. The converse is obvious.

(4) The functor $H \otimes \bullet\colon \mathcal{M}^H \to {}_H\mathcal{D}^H$ is a left adjoint and the canonical morphism given in (23) is an epimorphism. From [23, Lemma 2.7] it follows that $H \otimes \bullet$ preserves generators. ∎

3. The Drinfel'd double as a generalized smash product

The "usual" smash product $A\#H$ of an H-module algebra A and a Hopf algebra H (see [1] or [30]) can be generalized. In [13], Gamst and Hoechstman consider the smash product of an H-comodule algebra B and an H^*-comodule algebra C, in the situation where the Hopf algebra H is commutative, cocommutative and faithfully projective over the groundring k. Their main result is that $B\#C$ is an Azumaya algebra if B is an H-Galois object and C an H^*-Galois object in the sense of [6]. This generalized smash product may also be used to define the multiplication on Long's Brauer group of dimodule algebras, see [18]. Takeuchi [32] observed that, in the case where the Hopf algebra H is not necessarily faithfully projective, commutative or cocommutative, one can generalize the construction of [13] to define a smash product of an H-comodule algebra and an H-module algebra. It is this construction that we will be using in the sequel.

Let H be a Hopf algebra, A a left H-comodule algebra and D a left H-module algebra. We will use the following notation for the H-coaction on A and the H-action on D:

$$\rho_A(a) = \sum a_{[-1]} \otimes a_{[0]} \quad \text{and} \quad \psi_D(h \otimes d) = h \cdot d$$

for all $a \in A$, $d \in D$ and $h \in H$. The **smash product** of A and D is defined as follows: $A\#D = A \otimes D$ as k-module. $A\#D$ is a k-algebra, with multiplication given by the formula

$$(a\#d)(b\#e) = \sum a_{[0]}b\#d(a_{[-1]} \cdot e) \tag{24}$$

for all $a, b \in A$ and $d, e \in D$. It is straightforward to show that $A\#D$ is an associative algebra with unit $1_A\#1_D$ and that $i_A\colon A \to A\#D$, $a \mapsto a\#1_D$ and $i_D\colon D \to A\#D$, $d \mapsto 1_A\#d$ are algebra maps.

Remarks 3.1: (1) At first sight, our definition is different from the one in [32], where the following version of the smash product is introduced. $D\#'A$ is equal to $D \otimes A$ as a k-module, with multiplication given by the following formula:

$$(d\#'a)(e\#'b) = \sum d(a_{[-1]} \cdot e)\#'a_{[0]}b.$$

It is straightforward to show that τ: $A\#D \to D\#'A$, $\quad a\#d \mapsto d\#'a$ is an algebra isomorphism. If $A = H$, with the left H-comodule structure on A defined via Δ_H, then we recover the usual smash product from [1] and [30].

(2) Let A be a left H-comodule algebra and C a right H-module coalgebra with the structure map $C \otimes H \to H$, $c \otimes h \mapsto (c \leftharpoonup h)$. Then C^* can be given the structure of left H-module algebra as follows:

$$\langle h \cdot c^*, c\rangle = \langle c^*, c \leftharpoonup h\rangle$$

for all $h \in H$, $c \in C$ and $c^* \in C^*$. We can now consider the smash product $A\#C^*$. This is in fact the left-left version of the smash product considered by Doi in [11].

(3) Let D be a left H-comodule algebra, and A a twisted left H-module algebra in the sense of [2]. If σ: $H \otimes H \to A$ is a cocycle, then we can construct the **crossed product** $A\#_\sigma D$. As a module, the crossed product is equal to $A \otimes D$, and the multiplication is this time given by the formula

$$(a\#_\sigma d)(b\#_\sigma e) = \sum a(d_{[-2]} \cdot b)\sigma(d_{[-1]}, e_{[-1]})\#_\sigma d_{[0]}e_{[0]}.$$

Now let $G = (H, A, C)$ be a Drinfel'd datum. From (10), it follows that A is a left $H^{\rm op} \otimes H$-comodule algebra and that C is a right $H^{\rm op} \otimes H$-module coalgebra. The $H^{\rm op} \otimes H$-coaction on A is given by the formula

$$\rho(a) = \sum a_{[-1]} \otimes S^{-1}(a_{[1]}) \otimes a_{[0]}$$

for all $a \in A$, and the $H^{\rm op} \otimes H$-action on C is given by (11):

$$c \cdot (h \otimes k) = k \rightharpoonup c \leftharpoonup h$$

for all $c \in C$ and $h, k \in H$. By the above remarks, C^* can now be made into a left $H^{\rm op} \otimes H$-module algebra as follows:

$$\langle (h \otimes k) \triangleright c^*, c\rangle = \langle c^*, k \rightharpoonup c \leftharpoonup h\rangle. \tag{25}$$

We can now construct the smash product $A\#C^*$. We write $D_H(A,C) = A\#C^*$, and we call $D_H(A,C)$ the **Drinfel'd double** of the Drinfel'd datum $G = (H,A,C)$. $D_H(A,C)$ is an associative algebra with unit. In the next Proposition, we will show that, in the case $C = H = A$, we recover the Drinfel'd double $D(H)$ in the sense of Majid (cf. [19]).

PROPOSITION 3.2: *Let $G = (H,A,C)$ be a Drinfel'd datum. Then the multiplication rule in $D_H(A,C)$ can be written as follows:*

$$(a\#c^*)(b\#d^*) = \sum a_{[0]}b\#c^*\langle d^*, S^{-1}(a_{[1]})\rightharpoonup \bullet \leftharpoonup a_{[-1]}\rangle$$

for all $a,b \in A$ and $c^*, d^* \in C^*$. In particular, the algebras $D_H(H,H)$ and $D(H)$ are equal.

Proof: The formula follows immediately from the definition and (25):

$$\begin{aligned}(a\#c^*)(b\#d^*) &= \sum a_{[0]}b\#c^*[(a_{[-1]} \otimes S^{-1}(a_{[1]})) \triangleright d^*]\\ &= \sum a_{[0]}b\#c^*\langle d^*, S^{-1}(a_{[1]})\rightharpoonup \bullet \leftharpoonup a_{[-1]}\rangle. \qquad \blacksquare\end{aligned}$$

Remarks 3.3: (1) It follows from Proposition 3.2 that Majid's Drinfel'd double [20] is a generalized smash product in the sense of (24). The left $H \otimes H^{\mathrm{op}}$-coaction on H is given by the formula

$$h \mapsto \sum h_{(1)} \otimes S^{-1}(h_{(3)}) \otimes h_{(2)}$$

and the left $H \otimes H^{\mathrm{op}}$-action on H^* is given by the formula

$$\langle (h \otimes k) \triangleright h^*, l\rangle = \langle h^*, klh\rangle$$

for all $h,k,l \in H$ and $h^* \in H^*$. With these structure we obtain that

$$D(H) = H\#H^*$$

as a k-algebra.

(2) If H is finitely generated projective, then it follows from [22] that the k-algebras $D(H)$ and $\mathrm{End}_H^H(H \otimes H)$ are Morita equivalent. Here the crossed H-module structure on $H \otimes H$ is the one from Remark 2 preceding Corollary 2.10. Moreover, if H is Frobenius, then it can be shown that $D(H)$ and $\mathrm{End}_H^H(H \otimes H)$ are isomorphic (cf. [4]).

COROLLARY 3.4: *Let $G = (H, A, C)$ be a Drinfel'd datum and suppose that C is finitely generated and projective over k. Then the categories ${}^{C}\mathcal{D}(H)_A$ and $\mathcal{M}_{D_H(A,C)}$ are isomorphic.*

Proof: In the Remark following Corollary 2.4, we have seen that the categories ${}^{C}\mathcal{D}(H)_A$ and ${}^{C}\mathcal{M}(H \otimes H^{\rm op})_A$ are isomorphic. The fact that C is finitely generated and projective implies that the categories ${}^{C}\mathcal{M}(H \otimes H^{\rm op})_A$ and $\mathcal{M}_{A\#C^*}$ are isomorphic, cf. [11]. As we have seen above, the smash product $A\#C^*$ is nothing else than the Drinfel'd double $D_H(A, C)$, and the result follows. ∎

4. Appendix: A generalized smash coproduct

In [24], Molnar introduced the smash coproduct of a Hopf algebra H and an H-comodule coalgebra C. This is in fact a formal dual version of the usual smash product. Obviously, we can generalize this construction, to obtain the smash coproduct of an H-module coalgebra and an H-comodule coalgebra, and we can expect that this smash coproduct has properties that are similar to the properties of the smash product as discussed in Section 3. In this Section, we list these properties. The proofs are formal duals of the proofs of corresponding results for the smash product, and this is why we omitted them.

Let (C, ρ_C) be a right H-comodule coalgebra and (D, ψ_D) be a right H-module coalgebra, and denote

$$\rho_C(c) = \sum c_{\langle 0\rangle} \otimes c_{\langle 1\rangle} \quad \text{and} \quad \psi_D(d \otimes h) = dh$$

for all $c \in C$, $d \in D$ and $h \in H$. We define the **smash coproduct** of C and D as follows: $D \ltimes C = D \otimes C$ as a k-module, and the comultiplication and counit are given by the following formulas:

$$\Delta(d \ltimes c) = \sum (d_{(1)} \ltimes c_{(1)\langle 0\rangle}) \otimes (d_{(2)} c_{(1)\langle 1\rangle} \ltimes c_{(2)}), \tag{26}$$

$$\varepsilon(d \ltimes c) = \varepsilon_D(d)\varepsilon_C(c), \tag{27}$$

for all $d \in D$, $c \in C$. $D \ltimes C$ is a coassociative coalgebra with counit. The maps $\pi_D\colon D \ltimes C \to D\colon d \ltimes c \mapsto \varepsilon_C(c)d$ and $\pi_C\colon D \ltimes C \to C\colon d \ltimes c \mapsto \varepsilon_D(d)c$ are coalgebra maps, and $(\pi_D \otimes \pi_C) \circ \Delta = I_{D\ltimes C}$.

If C and D are finitely generated and projective as k-modules, then $(C^*\#D^*)^* \cong D \ltimes C$. If $D = H$ then $H \ltimes C$ is the usual smash coproduct defined in [24].

Remark 4.1: Let C be a right H-comodule coalgebra and D a right H-module coalgebra. In [7], the following type of smash coproduct is introduced: $C \times D = C \otimes D$, with comultiplication given by the formula

$$\Delta(c \times d) = \sum (c_{(1)} \times d_{(1)} S(c_{(2)\langle 1\rangle})) \otimes (c_{(2)\langle 0\rangle} \times d_{(2)}).$$

At first glance, this seems to be a different notion. However, if the antipode of H is bijective then our smash coproduct $D \ltimes C$ is isomorphic to the smash coproduct $C \times D$.

Now let C be a right H-comodule coalgebra and D a right H-module coalgebra. Following [3], we can introduce the category $\mathcal{M}(H)^{C,D}$. An object of $\mathcal{M}(H)^{C,D}$ is a k-module which is at once a right C-comodule (via $M \to M \otimes C$, $m \mapsto \sum m_{\{0\}} \otimes m_{\{1\}}$) and a right D-comodule (via $M \to M \otimes D$, $m \mapsto m_{[0]} \otimes m_{[1]}$) such that the following compatibility condition holds:

$$\sum m_{[0]\{0\}} \otimes m_{[0]\{1\}} \otimes m_{[1]} = \sum m_{\{0\}[0]} \otimes m_{\{1\}\langle 0\rangle} \otimes m_{\{0\}[1]} m_{\{1\}\langle 1\rangle} \tag{28}$$

for all $m \in M$.

Remark 4.2: Let G be a group. Recall from [27] the notion of G-graded coalgebra. This is a coalgebra C that can be written as a direct sum $C = \oplus_{\sigma \in G} C_\sigma$ of k-spaces such that

$$\Delta(c_\sigma) \subseteq \sum_{\lambda\mu=\sigma} C_\lambda \otimes C_\mu \quad \text{and} \quad \varepsilon(c_\sigma) = \delta_{\sigma 1}$$

for all $\sigma \in G$. It may be shown easily, cf. [8], that a G-graded coalgebra is nothing else than a right $k[G]$-comodule coalgebra. Now if G is a group and X is a right G-set, then we can introduce (see [7]) the notion of comodule graded by X. This is a right C-comodule M that can be written as a direct sum $M = \oplus_{x \in X} M_x$ such that

$$\rho(M_x) \subseteq \sum_{yg=x} M_y \otimes C_g$$

for all $x \in X$. The category of comodules graded by the G-set X is denoted by $\mathrm{gr}^{(G,X,C)}$, and it is easy to see that the categories $\mathrm{gr}^{(G,X,C)}$ and $\mathcal{M}(k[G])^{C,k[X]}$ are isomorphic.

Indeed, if $M \in \mathcal{M}(k[G])^{C,k[X]}$, then M is a right C-comodule and a right $k[X]$-comodule, hence M admits a decomposition $M = \oplus_{x \in G} M_x$. (cf. [25] for example). Take $m \in M$ homogeneous of degree x, i.e. $m \in M_x$. Then, it follows from (28) that

$$\sum m_{\{0\}} \otimes m_{\{1\}} \otimes x = \sum_{y\in X} \sum_{g\in G} m_{\{0\}y} \otimes m_{\{1\}g} \otimes yg,$$

hence

$$\rho(m) = \sum_{yg=x} m_{\{0\}y} \otimes m_{\{1\}g}, \quad \text{i.e.} \quad \rho(M_x) \subseteq \sum_{yg=x} M_x \otimes C_g,$$

and M is a comodule graded by X.

The following result is a generalization of [3, Proposition 1.3] and [7, Theorem 1.6].

PROPOSITION 4.3: *Let C be a right H-comodule coalgebra and D be a right H-module coalgebra. Then the categories $\mathcal{M}(H)^{C,D}$ and $\mathcal{M}^{D\ltimes C}$ are isomorphic.*

Proof: Take $M \in M^{D\ltimes C}$. We will write

$$\rho_M(m) = \sum m_{\langle 0\rangle} \otimes m_{\langle 1\rangle} \in M \otimes D \ltimes C$$

for all $m \in M$. Now let $\rho_M^C = (I_M \otimes \pi_C) \circ \rho_M$ and $\rho_M^D = (I_M \otimes \pi_D) \circ \rho_M$, that is

$$\rho_M^C(m) = \sum m_{\langle 0\rangle} \otimes \pi_C(m_{\langle 1\rangle}) = \sum m_{\{0\}} \otimes m_{\{1\}}$$

and

$$\rho_M^D(m) = \sum m_{\langle 0\rangle} \otimes \pi_D(m_{\langle 1\rangle}) = \sum m_{[0]} \otimes m_{[1]}.$$

From the fact that π_C and π_D are coalgebra maps, it follows that ρ_M^C and ρ_M^D define a C-coaction and a D-coaction on M. To show that $M \in \mathcal{M}(H)^{C,D}$, we have to show that (28) holds. For all $m \in M$, we have

$$\begin{aligned}
\rho_M(m) &= \sum m_{\langle 0\rangle} \otimes m_{\langle 1\rangle} \\
&= \sum m_{\langle 0\rangle} \otimes \pi_D(m_{\langle 1\rangle(1)}) \ltimes \pi_C(m_{\langle 1\rangle(2)}) \\
&= \sum m_{\langle 0\rangle\langle 0\rangle} \otimes \pi_D(m_{\langle 0\rangle\langle 1\rangle}) \ltimes \pi_C(m_{\langle 1\rangle}) \\
&= \sum m_{\{0\}\langle 0\rangle} \otimes \pi_D(m_{\{0\}\langle 1\rangle}) \ltimes m_{\{1\}} \\
&= \sum m_{\{0\}[0]} \otimes m_{\{0\}[1]} \ltimes m_{\{1\}}.
\end{aligned}$$

Hence we proved that

$$\rho(m) = \sum m_{\{0\}[0]} \otimes (m_{\{0\}[1]} \ltimes m_{\{1\}}). \tag{29}$$

Now we have that (M, ρ_M) is a $D \ltimes C$-comodule. This means

$$(\rho \otimes I)\rho(m) = (I \otimes \Delta)\rho(m)$$

or

$$\sum m_{\{0\}[0]\{0\}[0]} \otimes (m_{\{0\}[0]\{0\}[1]} \ltimes m_{\{0\}[0]\{1\}}) \otimes (m_{\{0\}[1]} \ltimes m_{\{1\}}) =$$
$$\sum m_{\{0\}[0]} \otimes (m_{\{0\}[1](1)} \ltimes m_{\{1\}(1)\langle 0\rangle}) \otimes (m_{\{0\}[1](2)} m_{\{1\}(1)\langle 1\rangle} \ltimes m_{\{1\}(2)}).$$

Now we apply $1 \otimes \pi_C \otimes \pi_D$ to both sides of this equation. We then obtain (28), and it follows that $M \in \mathcal{M}(H)^{C,D}$.

Conversely, let $M \in \mathcal{M}(H)^{C,D}$. Denote the C- and D-comodule structure maps as follows:

$$\rho_M^C(m) = \sum m_{\{0\}} \otimes m_{\{1\}} \quad \text{and} \quad \rho_M^D(m) = \sum m_{[0]} \otimes m_{[1]}.$$

Define

$$\rho_M \colon M \to M \otimes (D \ltimes C)$$

by

$$\rho_M(m) = \sum m_{\{0\}[0]} \otimes (m_{\{0\}[1]} \ltimes m_{\{1\}}).$$

A technical but straightforward computation shows that $M \in M^{D \ltimes C}$. This finishes our proof. ∎

We can now give the relation between the smash coproduct and the category of Doi–Hopf modules. Suppose that B is a left H-comodule algebra, and that D is a right H-module coalgebra. ${}_B\mathcal{M}(H)^D$ will be the category of k-modules that are left B-modules and right D-comodules such that the following compatibility relation holds:

$$\rho_M(bm) = \sum b_{[0]} m_{[0]} \otimes m_{[1]} S^{-1}(b_{[-1]}) \tag{30}$$

for all $b \in B$ and $m \in M$.

PROPOSITION 4.4: *Suppose that the antipode S of the Hopf algebra H is bijective. Let C be a right H-comodule coalgebra and D a right H-module coalgebra. If C is finitely generated projective as a k-module, then the categories ${}_{C^*}\mathcal{M}(H)^D$ and $\mathcal{M}^{C,D}$ are isomorphic.*

Proof: Recall that C^* is a left H-comodule algebra. Now take $M \in {}_{C^*}\mathcal{M}^D$. We can define a right coaction of C on M as follows: $\rho_M^C(m) = \sum m_{\{0\}} \otimes m_{\{1\}}$ if and only if $c^*.m = \sum \langle c^*, m_{\{1\}}\rangle m_{\{0\}}$ for all $c^* \in C^*$. Let us show that, with this C-coaction and the original D-coaction, M is an object of $\mathcal{M}(H)^{C,D}$. Using (30), we obtain that

$$\begin{aligned}\sum \langle c^*, m_{\{1\}}\rangle m_{\{0\}[0]} \otimes m_{\{0\}[1]} \\ &= \sum \langle c^*_{\langle 0\rangle}, m_{[0]\{1\}}\rangle m_{[0]\{0\}} \otimes m_{[1]}S^{-1}(c^*_{\langle 1\rangle}) \\ &= \sum \langle c^*, m_{[0]\{1\}\langle 0\rangle}\rangle m_{[0]\{0\}} \otimes m_{[1]}S^{-1}(m_{[0]\{1\}\langle 1\rangle})\end{aligned}$$

and therefore

$$\sum m_{\{0\}[0]} \otimes m_{\{0\}[1]} \otimes m_{\{1\}} = \sum m_{[0]\{0\}} \otimes m_{[1]}S^{-1}(m_{[0]\{1\}\langle 1\rangle}) \otimes m_{[0]\{1\}\langle 0\rangle}.$$

Apply ρ_C to the last factor. Then let the fifth factor act on the second one. This yields

$$\begin{aligned}\sum m_{\{0\}[0]} \otimes m_{\{0\}[1]}m_{\{1\}\langle 0\rangle} \otimes m_{\{1\}\langle 0\rangle} \\ &= \sum m_{[0]\{0\}} \otimes m_{[1]}S^{-1}(m_{[0]\{1\}\langle 2\rangle})m_{[0]\{1\}\langle 1\rangle} \otimes m_{[0]\{1\}\langle 0\rangle} \\ &= \sum m_{[0]\{0\}} \otimes m_{[1]} \otimes m_{[0]\{1\}}\end{aligned}$$

and (28) follows. In a similar way, we can show that every object in $\mathcal{M}(H)^{C,D}$ is also an object of ${}_{C^*}\mathcal{M}^D$. ∎

Similar properties hold for the category of left $D \ltimes C$-comodules. As above, let C be a right H-comodule coalgebra and D a right H-module coalgebra. An object of the category ${}^{C,D}\mathcal{M}(H)$ is a k-module that is at once a left C-comodule and a left D-comodule such that the following compatibility relation holds:

$$\begin{aligned}\sum m_{\{-1\}} \otimes m_{\{0\}[-1]} \otimes m_{\{0\}[0]} \\ =\sum m_{[0]\{-1\}\langle 0\rangle} \otimes m_{[-1]}m_{[0]\{-1\}\langle 1\rangle} \otimes m_{[0]\{0\}}\end{aligned} \tag{31}$$

for all $m \in M$. Here the D-coaction and the C-coaction on M are denoted as follows:

$$\rho^C_M(m) = \sum m_{\{-1\}} \otimes m_{\{0\}} \quad \text{and} \quad \rho^D_M(m) = \sum m_{[-1]} \otimes m_{[0]}.$$

With these notations, we have the following result. The proof is omitted, since it is similar to the proofs of Propositions 4.3 and 4.4.

PROPOSITION 4.5: *Let C be a right H-comodule coalgebra and D a right H-module coalgebra. Then the categories ${}^{D\ltimes C}\mathcal{M}$ and ${}^{C,D}\mathcal{M}(H)$ are isomorphic. If C is finitely generated projective as a k-module, then they are also isomorphic to ${}^D\mathcal{M}(H)_{C^*}$.*

Let us now apply the above results to crossed H-modules. Suppose that C is an H-bimodule coalgebra and that A is an H-bicomodule algebra. As we have already seen, C is a right $H \otimes H^{\rm op}$-module coalgebra and A is a left $H \otimes H^{\rm op}$-comodule algebra, the structure maps are given by equations (10) and (11). Now suppose that A is finitely generated and projective as a k-module. Then A^* is a right $H \otimes H^{\rm op}$-comodule coalgebra, and the coaction is given by

$$\rho(a^*) = \sum a^*_{[0]} \otimes a^*_{[1]} \otimes S^{-1}(a^*_{[-1]})$$

if and only if

$$\sum \langle a^*_{[0]}, a\rangle a^*_{[1]} \otimes a^*_{[-1]} = \sum \langle a^*, a_{[0]}\rangle a_{[-1]} \otimes a_{[1]} \tag{32}$$

for all $a \in A$. From Proposition 4.5 and the remark preceding Proposition 2.6, it follows that

$$^{C}\mathcal{D}(H)_A \cong {}^{C}\mathcal{M}(H \otimes H^{\rm op})_A \cong {}^{C \ltimes A^*}\mathcal{M}.$$

We define the **Drinfel'd codouble** of A and C as the smash coproduct $D^*_H(A, C) = C \ltimes A^*$. It is then easy to show that the comultiplication on $D^*_H(A, C)$ is given by the formula

$$\Delta(c \ltimes a^*) = \sum (c_{(1)} \ltimes a^*_{(1)[0]}) \otimes (S^{-1}(a^*_{(1)[-1]}) \rightharpoonup c_{(2)} \leftharpoonup a^*_{(1)[1]}) \ltimes a^*_{(2)}). \tag{33}$$

It follows that $D^*_H(A, C) \cong D_H(A, C)^*$ if A and C are both finitely generated and projective. Therefore, in the case where H is finitely generated and projective, (33) gives an explicit formula for the comultiplication on the dual of the Drinfel'd double $D(H)$. This comultiplication has been described earlier by Majid in [21].

We will now describe induction functors connecting categories of (C, D)-comodules. The situation is similar to the one encountered in Theorem 1.1.

Consider two threetuples (H, C, D) and (H', C', D'), where H and H' are Hopf algebras, C (resp. C') is a right H (resp. H')-comodule algebra and D (resp. D') is a right H (resp. H')-module coalgebra. Let $F = (f, u, v)$: $(H, C, D) \to (H', C', D')$ be such that f: $H \to H'$ is a Hopf algebra map, u: $C \to C'$ is a right H'-comodule coalgebra map (C is right H'-comodule via f) and v: $D \to D'$ is a right H-module coalgebra map (D' is right H-module via f). Then it is straightforward to show that the map

$$\varphi: D \ltimes C \to D' \ltimes C': d \ltimes c \mapsto v(d) \ltimes u(c) \tag{34}$$

is comultiplicative. Now consider the corestriction of coscalars functor φ_* : $\mathcal{M}^{D\ltimes C} \to \mathcal{M}^{D'\ltimes C'}$. From [10, Proposition 6], it follows that the functor φ_* has a right adjoint

$$\bullet\Box_{D'\ltimes C'} D \ltimes C\colon \mathcal{M}^{D'\ltimes C'} \mapsto M^{D\ltimes C}.$$

Using Proposition 4.3, we therefore obtain a functor

$$F_*\colon \mathcal{M}(H)^{C,D} \to \mathcal{M}(H')^{C',D'}$$

which has a right adjoint. This extends Theorem 2.1 in [7].

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